Harmonic bundle solutions of Topological-antitopological fusion in para-complex geometry

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Abstract

In this work we introduce the notion of a para-harmonic bundle, i.e. the generalization of a harmonic bundle [Si] to para-complex differential geometry. We show that para-harmonic bundles are solutions of the para-complex version of metric tt*-bundles introduced in [Sch2]. Further we analyze the correspondence between metric para-tt^*-bundles of rank $2r$ over a para-complex manifold M and para-pluriharmonic maps from M into the pseudo-Riemannian symmetric space $\mathrm{GL}(r,\mathbb{R})/\mathrm{O}(p,q)$, which was shown in [Sch2], in the case of a para-harmonic bundle. It is proven, that for para-harmonic bundles the associated para-pluriharmonic maps take values in the totally geodesic subspace $\mathrm{GL}(r,C)/\mathrm{U}^{\pi}(C^r)$ of $\mathrm{GL}(2r,\mathbb{R})/\mathrm{O}(r,r)$. This defines a map Φ from para-harmonic bundles over M to para-pluriharmonic maps from M to $\mathrm{GL}(r,C)/\mathrm{U}^{\pi}(C^r)$. The image of Φ is also characterized in the paper.

Keywords: para-complex geometry, para-harmonic bundles, para-tt^*-geometry, para-pluriharmonic maps and pseudo-Riemannian symmetric spaces

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1 Introduction

Para-complex geometry was first introduced in 1952 by P. Libermann [L]. For a survey on this subject we refer to [CFG]. Examples can be obtained from para-hermitian symmetric spaces, see [B]. These spaces provide examples of para-Kähler manifolds, which are also named bi-Lagrangian manifolds (cf. [EST] for a survey). We remark that para-Kähler manifolds are forced to have a metric of split signature and that consequentely the later considered harmonic and para-pluriharmonic maps with a para-Kähler manifold as source-manifold are no longer solutions of an elliptic equation.

The complex notion of tt^*-geometries originates in the physics of topological field-theories (see [CV]). Motivated by the recent study of the geometries appearing in Euclidean supersymmetry by [CMMS] we generalized in [Sch2] the notion of tt^*-bundles to para-complex geometry, which we denoted para-tt^*-bundles. We showed that special para-complex and special para-Kähler manifolds are solutions of tt^*-bundles. Further we roughly gave a correspondence between metric para-tt^*-bundles (E, D, S, g) and para-pluriharmonic maps into the pseudo-Riemannian symmetric space $\mathrm{GL}(r, \mathbb{R})/\mathrm{O}(p, q)$ where (p, q) is the signature of the bundle metric g on the vector bundle E. In the case of a para-tt^*-bundle induced by a special para-Kähler manifold this para-pluriharmonic map was related to the dual Gauß map. Analogous results were shown in [CS, Sch1] for complex geometry.

Another class of tt^*-bundles in the complex setting are harmonic bundles with positive definite hermitian metric which were introduced by Simpson [Si]. In the same paper Simpson gave a correspondence between harmonic bundles over a compact Kähler manifold M and harmonic maps from M into $\mathrm{GL}(r, \mathbb{C})/U(r)$. In [Sch3] we related this result to the complex version of the above mentioned correspondence between metric tt^*-bundles and pluriharmonic maps. In fact, we were able to obtain a more general result by applying our correspondence to harmonic bundles with possible indefinite metric over an arbitrary complex manifold. Simpsons result was recovered from this correspondence.

Against this background it arises the question if there exists a notion of harmonic bundle in the para-complex setting, if this generalization of a harmonic bundle supplies a solution of para-tt^*-geometry and finally if there exists a correspondence of these generalized harmonic bundles and para-pluriharmonic maps from M into the pseudo-Riemannian symmetric space $\mathrm{GL}(r, C)/\mathrm{U}^\pi(C^r)$, which is the para-complex analogue of $\mathrm{GL}(r, \mathbb{C})/U(r)$. In this paper we answer positively to these questions.
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2 Para-complex differential geometry

We shortly recall some notions and facts of para-complex differential geometry. The idea of para-complex geometry is to replace the complex structure J with $J^2 = -\mathbb{1}$ (on a finite dimensional vector space V) by the para-complex structure $\tau \in End(V)$ satisfying $\tau^2 = \mathbb{1}$ such that the ± 1-eigenspaces have the same dimension. An almost para-complex structure

on a smooth manifold M is an endomorphism-field τ, which is a point-wise para-complex structure. If the eigen-distubutions $T^{\pm}M$ are integrable τ is called para-complex structure on M and M is called a para-complex manifold. Analogous to complex geometry there exists a tensor, also called Nijenhuis tensor, which is the obstruction to the integrability of the para-complex structure.
The real algebra, which is generated by 1 and by the para-complex unit e with $e^2 = 1$, is called the para-complex numbers and denoted by C. For all $z = x + ey \in C$ with $x, y \in \mathbb{R}$ we define the para-complex conjugation as $\bar{\cdot} : C \to C, x + ey \mapsto x - ey$ and the real and imaginary parts of z by $\Re(z) := x,\ \Im(z) := y$. The free C-module C^n is a para-complex vector space where its para-complex structure is just the multiplication with e and the para-complex conjugation of C extends to $\bar{\cdot} : C^n \to C^n$, $v \mapsto \bar{v}$.
Note, that $z\bar{z} = x^2 - y^2$. Therefore the algebra C is sometimes called the hypercomplex numbers. The circle $\mathbb{S}^1 = \{z = x+iy \in \mathbb{C} \,|\, x^2+y^2 = 1\}$ is replaced by the four hyperbolas $\{z = x + ey \in C \,|\, x^2 - y^2 = \pm 1\}$. We define $\tilde{\mathbb{S}}^1$ to be the hyperbola given by the one parameter group $\{z(\theta) = \cosh(\theta) + e \sinh(\theta) \,|\, \theta \in \mathbb{R}\}$.
A para-complex vector space (V, τ) endowed with a pseudo-Euclidean metric g is called para-hermitian vector space, if g is τ-anti-invariant, i.e. $\tau^* g = -g$. The para-unitary group of V is defined as the group of automorphisms

$$U^\pi(V) := \mathrm{Aut}(V, \tau, g) := \{L \in GL(V) | [L, I] = 0 \text{ and } L^* g = g\}$$

and its Lie-algebra is denoted by $\mathfrak{u}^\pi(V)$. For $C^n = \mathbb{R}^n \oplus e\mathbb{R}^n$ the standard para-hermitian structure is defined by the above para-complex structure and the metric $g = \mathrm{diag}(\mathbb{1}, -\mathbb{1})$ (cf. Example 7 of [CMMS]). The corresponding para-unitary group is given by (cf. Proposition 2 of [CMMS]):

$$U^\pi(C^n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} | \, A, B \in \mathrm{End}(\mathbb{R}^n), A^T A - B^T B = \mathbb{1}_n,\ A^T B - B^T A = 0 \right\}.$$

There exist two bi-gradings on the exterior algebra: The one is induced by the splitting in $T^{\pm}M$ and denoted by $\Lambda^k T^* M = \bigoplus_{k=p+q} \Lambda^{p+,q-} T^* M$ and induces an obvious bi-grading on exterior forms with values in a vector bundle E. The second is induced by the decomposition of the para-complexified tangent bundle $TM^C = TM \otimes_{\mathbb{R}} C$ into the subbundles $T_p^{1,0}M$ and $T_p^{0,1}M$ which are defined as the $\pm e$-eigenbundles of the para-complex linear extension of τ. This induces a bi-grading on the C-valued exterior forms noted $\Lambda^k T^* M^C = \bigoplus_{k=p+q} \Lambda^{p,q} T^* M$ and finally on the C-valued differential forms on M $\Omega^k_C(M) = \bigoplus_{k=p+q} \Omega^{p,q}(M)$. In the case $(1, 1)$ and $(1+, 1-)$ the two gradings induced by τ coincide, in the sense that $\Lambda^{1,1} T^* M = (\Lambda^{1+,1-} T^* M) \otimes C$. The bundles $\Lambda^{p,q} T^* M$ are para-complex vector bundles in the following sense: A para-complex vector bundle of rank r over a para-complex manifold (M, τ) is a smooth real vector bundle $\pi : E \to M$ of rank $2r$ endowed with a fiberwise para-complex structure $\tau^E \in \Gamma(\mathrm{End}\,(E))$. We denote it by (E, τ^E). In the following text we always identify the fibers of a para-complex vector bundle E of rank r with the free C-module C^r.
One has a notion of para-holomorphic vector bundles, too. These were extensively studied in a common work with M.-A. Lawn-Paillusseau [LS].

3 Para-pluriharmonic maps

In this section we discuss para-pluriharmonic maps, which are the analogue of pluriharmonic maps in para-complex geometry.
Throughout this section we consider a para-complex manifold (M, τ). Given a pseudo-Riemannian manifold (N, h) a map $f : M \to N$ is called para-pluriharmonic if $f|_C$ is harmonic for every para-complex curve $C \subset M$, i.e. for every real two-dimensional para-complex submanifold. We remark, that the harmonicity of $f|_C$ is independent of the choice of a pseudo-Riemannian metric in the conformal class of C.
Let us further fix an adapted connection D on (M, τ), i.e. a connection which satisfies

$$D_{\tau Y} X = \tau D_Y X \tag{3.1}$$

for all vector fields X, Y with $\mathcal{L}_X \tau = 0$ (i.e. for which $X + e\tau X$ is para-holomorphic). The notion of adapted connections on para-holomorphic vector bundles over para-complex manifolds was introduced in [LS]. The existence of adapted connections on para-complex manifolds was assured in proposition 9 of [Sch2].
A map $f : M \to N$ is para-pluriharmonic if and only if it satisfies the following equation

$$(\nabla df)^{(1,1)} = 0, \quad df \in \Gamma(T^*M \otimes f^*TN), \tag{3.2}$$

where ∇ is induced by the adapted connection D and the Levi-Civita connection of h.
Later, we use a special class of para-pluriharmonic morphisms:

Proposition 1 *Let (M, τ) be a para-complex manifold, X, Y be pseudo-Riemannian manifolds and $\Psi : X \to Y$ be a totally geodesic immersion. Then a map $f : M \to X$ is para-pluriharmonic if and only if $\Psi \circ f : M \to Y$ is para-pluriharmonic.*

Proof: It is well-known (cf. Eells and Sampson [ES]) that a map $f : M \to X$ is harmonic if and only if $\Psi \circ f : M \to Y$ is harmonic. The definition of para-pluriharmonic maps finishes the proof. □

We are going to apply this result to the symmetric spaces G/K with $G = GL(2r, \mathbb{R})$ and $K = O(r, r)$ or $G = GL(r, C)$ and $K = U^\pi(C^r)$. We discuss this for the second example, since the first can be found in [Sch2] and is very similar.

In the following subsection we identify C^r with $\mathbb{R}^r \oplus e\mathbb{R}^r = \mathbb{R}^{2r}$. The multiplication with e equals the automorphism $E = \begin{pmatrix} 0 & \mathbb{1}_r \\ \mathbb{1}_r & 0 \end{pmatrix}$ and $GL(r, C)$ (respectively $\mathfrak{gl}_r(C)$) consists of the elements in $GL(2r, \mathbb{R})$ (respectively $\mathfrak{gl}_{2r}(\mathbb{R})$) commuting with E.
First, we have to discuss some linear algebra: We start with the notion of para-hermitian sesquilinear scalar products on para-complex vector spaces:

Definition 1

1. *A* para-hermitian sesquilinear scalar product *is a non-degenerate sesquilinear form $h : C^r \times C^r \to C$, i.e. (i) h is non-degenerate: Given $w \in C^r$ such that for all $v \in C^r$ it holds $h(v, w) = 0$, then it follows $w = 0$, (ii) $h(v, w) = \overline{h(w, v)}$, $\forall\, v, w \in C^r$, (iii) $h(\lambda v, w) = \lambda h(v, w)$, $\forall\, \lambda \in C;\ v, w \in C^r$.*

2. *Let* $z = (z^1, \ldots, z^r)$ *and* $w = (w^1, \ldots, w^r)$ *be two elements of* C^r, *then one defines the* standard C-bilinear scalar product *on* C^r *by* $z \cdot w := \sum_{i=1}^{r} z^i w^i$ *and the* standard para-hermitian sesquilinear scalar product *by* $(z, w)_{C^r} := z \cdot \bar{w}$.

3. *Given a matrix* C *of* $End(C^r) = End_C(C^r)$, *we define the* para-hermitian conjugation *by* $C \mapsto C^h = \bar{C}^t$. *We call* C para-hermitian *if and only if* $C^h = C$. *We denote by* $herm(C^r)$ *the set of para-hermitian endomorphisms and by* $Herm(C^r) = herm(C^r) \cap GL(r, C)$.

We remark, that there is **no** notion of para-hermitian signature, since from $h(v, v) = -1$ for a $v \in C^r$ we obtain $h(ev, ev) = 1$.

Proposition 2

(a) *Given an element* C *of* $End(C^r)$ *then it holds* $(Cz, w)_{C^r} = (z, C^h w)_{C^r}$, $\forall z, w \in C^r$.

(b) *The set* $herm(C^r)$ *is a real vector space.*

(c) *There is a bijective correspondence between* $Herm(C^r)$ *and para-hermitian sesquilinear scalar products* h *on* C^r *given by* $H \mapsto h(\cdot, \cdot) := (H\cdot, \cdot)_{C^r}$.

We now analyze the map which corresponds to taking the real part $\mathrm{Re}\, h$ of h. This is the map $\mathcal{R}$ satisfying $\mathrm{Re}\, h = (\mathcal{R}(H)\cdot, \cdot)_{\mathbb{R}^{2r}}$, where $(\cdot, \cdot)_{\mathbb{R}^{2r}}$ is the Euclidean standard scalar product on $\mathbb{R}^{2r}$.
First we identify some usual operations in the above identification $C^r = \mathbb{R}^r \oplus e\mathbb{R}^r = \mathbb{R}^{2r}$: An endomorphism $C \in \mathrm{End}(C^r)$ decomposes in its real part A and its imaginary part B, i.e. $C = A + eB$ where $A, B \in \mathrm{End}(\mathbb{R}^r)$ and C is identified via a map, which we denote by ι, with the matrix

$$\iota(C) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

The para-complex conjugation of C, i.e. $\bar{C} = A - eB$, and the transposition $C^t = A^t + eB^t$ correspond to

$$\iota(\bar{C}) = \begin{pmatrix} A & -B \\ -B & A \end{pmatrix} \text{ and } \iota(C^t) = \begin{pmatrix} A^t & B^t \\ B^t & A^t \end{pmatrix} = \iota(C)^T,$$

where $\cdot^T$ is the transposition in $\mathrm{End}(\mathbb{R}^{2r})$ and the adjoint with respect to $(\cdot, \cdot)_{C^r}$ is $C^h = \bar{C}^t$ which corresponds to

$$\iota(C^h) = \begin{pmatrix} A^t & -B^t \\ -B^t & A^t \end{pmatrix} \stackrel{(*)}{=} \mathbb{1}_{r,r}\, \iota(C)^T\, \mathbb{1}_{r,r}.$$

The equality in $(*)$ is due to the calculation:

$$\begin{aligned} \iota(C^h)\mathbb{1}_{r,r} &= \begin{pmatrix} A^t & -B^t \\ -B^t & A^t \end{pmatrix} \mathbb{1}_{r,r} = \begin{pmatrix} A^t & B^t \\ -B^t & -A^t \end{pmatrix} \\ &= \mathbb{1}_{r,r} \begin{pmatrix} A^t & B^t \\ B^t & A^t \end{pmatrix} = \mathbb{1}_{r,r}\, \iota(C)^T = \mathbb{1}_{r,r}\, \iota(C^t) \end{aligned} \tag{3.3}$$

with

$$\mathbb{1}_{r,r} = \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & -\mathbb{1}_r \end{pmatrix}.$$

A para-hermitian sesquilinear scalar product h corresponds to a para-hermitian matrix $H \in \mathrm{Herm}(C^r)$ (compare with proposition 2) defined by $h(\cdot,\cdot) = (H\cdot,\cdot)_{C^r}$. The condition $C^h = C$, i.e. C is para-hermitian, means in our model that $C = A + eB$ such that the real matrices A and B satisfy $A = A^t$ and $B = -B^t$.
The group $GL(r, C)$ operates on $\mathrm{Herm}(C^r)$ via

$$GL(r, C) \times \mathrm{Herm}(C^r) \to \mathrm{Herm}(C^r), \quad (g, B) \mapsto g \cdot B := (g^{-1})^h B g^{-1}.$$

Let us denote by $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$ the symmetric $r \times r$ matrices of symmetric signature (p, q) with $r = p + q$. We consider the $GL(r, \mathbb{R})$-action on $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$ given by $(g^{-1}, S) \mapsto g^{-1} \cdot S = g^T S g$ with $g \in GL(r, \mathbb{R})$ and $S \in \mathrm{Sym}_{p,q}(\mathbb{R}^r)$.

Lemma 1 *The map $\mathcal{R}$ is given by $\mathcal{R} = \mathbb{1}_{r,r} \circ \iota$ and it defines an equivariant immersion $\mathcal{R} : Herm(C^r) \to Sym_{r,r}(\mathbb{R}^{2r})$.*

Proof: First we determine $\mathcal{R}$: With $z, w \in C^r$ we have $\beta(z, w) := \mathrm{Re}\,(z, w)_{C^r} = \frac{1}{2}(z \cdot \bar{w} + \bar{z} \cdot w)$ and $\mathrm{Re}\, h(z, w) = \mathrm{Re}\,(Hz, w)_{C^r} = \frac{1}{2}\left[(Hz) \cdot \bar{w} + (\overline{Hz}) \cdot w\right] = \beta(Hz, w)$. Further we remark that $\beta(\cdot,\cdot) = \mathrm{Re}\,(\cdot,\cdot)_{C^r} = (\cdot,\cdot)_{\mathbb{R}^{r,r}}$, where $(\cdot,\cdot)_{\mathbb{R}^{r,r}} = (\mathbb{1}_{r,r}\cdot,\cdot)_{\mathbb{R}^{2r}}$ is the (pseudo-)Euclidean standard scalar product of signature (r, r) on $\mathbb{R}^{2r}$. This yields $\mathrm{Re}\, h(z, w) = (Hz, w)_{\mathbb{R}^{r,r}} = (\mathbb{1}_{r,r} Hz, w)_{\mathbb{R}^{2r}}$ and for $H = A + eB$ with $A, B \in \mathrm{End}(\mathbb{R}^r)$

$$\mathcal{R}(H) = \mathbb{1}_{r,r}\iota(H) = \mathbb{1}_{r,r} \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}.$$

Since H is para-hermitian, we obtain $\mathcal{R}(H)^T = \mathcal{R}(H)$. The symmetric signature of the symmetric matrix $\mathcal{R}(H)$ is (r, r), as it is the real part of a para-hermitian sesquilinear scalar product. Summarizing we have $\mathcal{R} : \mathrm{Herm}(C^r) \to \mathrm{Sym}_{r,r}(\mathbb{R}^{2r}), \quad H \mapsto \mathcal{R}(H) = \mathbb{1}_{r,r}\iota(H)$. Moreover one sees, that the map $\mathcal{R}$ has maximal rank.
We show, that $\mathcal{R}$ is equivariant with respect to the above mentioned actions. In fact,

$$\begin{aligned} \mathcal{R}(g^{-1} \cdot H) &= \mathcal{R}(g^h H g) = \mathbb{1}_{r,r}\,\iota(g^h H g) \\ &= \mathbb{1}_{r,r}\,\iota(g^h)\,\iota(H)\,\iota(g) \overset{(3.3)}{=} \iota(g)^T\,\mathbb{1}_{r,r}\,\iota(H)\,\iota(g) \\ &= \iota(g)^T\,\mathcal{R}(H)\,\iota(g) = \iota(g)^{-1} \cdot \mathcal{R}(H). \end{aligned}$$

□

Lemma 2 *The inclusion $GL(r, C)/U^\pi(C^r) \hookrightarrow GL(2r, \mathbb{R})/O(r, r)$ is totally geodesic.*

Proof: The decomposition $\mathfrak{gl}_{2r}(\mathbb{R}) = \mathrm{sym}_{r,r}(\mathbb{R}^{2r}) \oplus \mathfrak{o}(r, r)$, where $\mathrm{sym}_{r,r}(\mathbb{R}^{2r})$ are the symmetric matrices with respect to $(\cdot,\cdot)_{\mathbb{R}^{r,r}}$, is a symmetric decomposition associated to the symmetric space $GL(2r, \mathbb{R})/O(r, r)$ and hence

$$[[\mathrm{sym}_{r,r}(\mathbb{R}^{2r}), \mathrm{sym}_{r,r}(\mathbb{R}^{2r})], \mathrm{sym}_{r,r}(\mathbb{R}^{2r})] \subset \mathrm{sym}_{r,r}(\mathbb{R}^{2r}).$$

Let $A, B, C \in \mathrm{herm}(C^r)$. From $[A, E] = [B, E] = [C, E] = 0$, we conclude with the Jacobi identity $[[A, B], E] = 0$ and $[[[A, B], C], E] = 0$. Hence $T_{\mathbb{1}_r} GL(r, C)/U^\pi(C^r) = \mathrm{herm}(C^r)$ is a Lie-triple-system in $T_{\mathbb{1}_{r,r}} \mathrm{Sym}_{r,r}(\mathbb{R}^{2r}) = \mathrm{sym}_{r,r}(\mathbb{R}^{2r})$, i.e.

$$[[\mathrm{herm}(C^r), \mathrm{herm}(C^r)], \mathrm{herm}(C^r)] \subset \mathrm{herm}(C^r)$$

and consequently $GL(r, C)/U^\pi(C^r)$ is a totally geodesic submanifold of $GL(2r, \mathbb{R})/O(r, r)$. □

Lemma 3 *The $GL(r, C)$-action on $Herm(C^r)$ induces a diffeomorphism*

$$\begin{aligned} \Psi \ : \ & GL(r, C)/U^\pi(C^r) \tilde{\to} Herm(C^r) \subset GL(r, C), \\ & gU^\pi(C^r) \mapsto g \cdot \mathbb{1}_r = (g^{-1})^h \mathbb{1}_r g^{-1} = (g^{-1})^h g^{-1}. \end{aligned} \tag{3.4}$$

Proof: The stabilizer of $\mathbb{1}_r$ under the $GL(r, C)$-action on $\mathrm{Herm}(C^r)$ is shown by a short calculation to be $GL(r, C)_{\mathbb{1}_r} = \{g \in GL(r, C) \,|\, g \cdot \mathbb{1}_r = (g^{-1})^h \mathbb{1}_r g^{-1} = \mathbb{1}_r\} = U^\pi(C^r)$.
If the operation $\cdot$ is transitive we obtain, by the orbit stabilizer theorem, a diffeomorphism

$$\begin{aligned} \Psi \ : \ & GL(r, C)/U^\pi(C^r) \tilde{\to} \mathrm{Herm}(C^r) \subset GL(r, C), \\ & gU^\pi(C^r) \mapsto g \cdot \mathbb{1}_r = (g^{-1})^h \mathbb{1}_r g^{-1} = (g^{-1})^h g^{-1}. \end{aligned} \tag{3.5}$$

The transitivity is due to the following argument: Any para-hermitian sesquilinear scalar product is uniquely determined by its real part, which lies in $\mathrm{Sym}_{r,r}(\mathbb{R}^{2r})$. On this space $GL(2r, \mathbb{R})$ acts transitively.
We claim: $h' = g \cdot h$ with some para-hermitian sesquilinear scalar product h and an element $g \in GL(2r, \mathbb{R})$ is a para-hermitian sesquilinear scalar product if and only if $g \in GL(r, C)$.
Proof: This claim follows from a short calculation: Let $v, w \in C^r$ and $\lambda \in C$: On the one hand it holds $h'(\lambda v, w) = \lambda h'(v, w) = \lambda (g \cdot h)(v, w) = h(\lambda g^{-1} v, g^{-1} w)$ and on the other hand $h'(\lambda v, w) = (g \cdot h)(\lambda v, w) = h(g^{-1} \lambda v, g^{-1} w)$. Subtracting these two equations yields $h((g^{-1} \lambda - \lambda g^{-1}) v, g^{-1} w) = 0$. Setting $w = gw'$ with arbitrary $w' \in C^r$ we obtain $h((g^{-1} \lambda - \lambda g^{-1}) v, w') = 0$. Since g is invertible and h is non-degenerate, we conclude $g^{-1} \lambda v = \lambda g^{-1} v$, which implies the C-linearity of g. □

We are now going to analyze para-pluriharmonic maps into these spaces:

Proposition 3 *Let (M, τ) be a para-complex manifold and endow $GL(r, C)/U^\pi(C^r)$ with the (pseudo-)metric induced by the trace-form on $GL(r, C)$. Then the map $\Psi : GL(r, C)/U^\pi(C^r) \tilde{\to} Herm(C^r)$ defined in equation (3.4) is totally geodesic and a map $\phi : M \to GL(r, C)/U^\pi(C^r)$ is para-pluriharmonic if and only if*

$$\psi = \Psi \circ \phi \, : \, M \to GL(r, C)/U^\pi(C^r) \tilde{\to} Herm(C^r) \subset GL(r, C)$$

is para-pluriharmonic.

Proof: To prove this we define $\sigma \, : \, GL(r, C) \to GL(r, C), \quad g \mapsto (g^{-1})^h$. The map σ is a homomorphism and an involution satisfying $GL(r, C)^\sigma = U^\pi(C^r)$. Hence the Cartan immersion can be written as

$$i \, : \, GL(r, C)/U^\pi(C^r) \to GL(r, C), \quad g \mapsto g\sigma(g^{-1}) = gg^h = gg^h = \Psi \circ \Lambda(g),$$

where Λ is the map induced on $GL(r,C)/U^\pi(C^r)$ by $\tilde{\Lambda} : GL(r,C) \to GL(r,C)$, $g \mapsto (g^{-1})^h$ which is an isometry of the invariant metric, since $g \mapsto g^h = \mathbb{1}_{r,r} g^T \mathbb{1}_{r,r}$ and $g \mapsto g^{-1}$ are isometries of the invariant metric. Therefore Ψ is totally geodesic, since i is totally geodesic. Proposition 1 finishes the proof. $\square$

To be complete we mention the related symmetric decomposition:

$$\mathfrak{h} = \{A \in \mathfrak{gl}_r(C) \mid A^h = -A\} = \mathfrak{u}^\pi(C^r) \text{ and } \mathfrak{p} = \{A \in \mathfrak{gl}_r(C) \mid A^h = A\} = \mathrm{herm}(C^r).$$

Let $\tilde{\Psi} : GL(r,\mathbb{R})/O(p,q) \tilde{\to} \mathrm{Sym}_{p,q}(\mathbb{R}^r) \subset GL(r,\mathbb{R})$ be the identification obtained from the above mentioned action of $GL(r,\mathbb{R})$ on $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$. With a similar argumentation one can prove (compare [Sch2]) that $\tilde{\Psi}$ is a totally geodesic immersion. Summarizing the above information and the results of [Sch2], we have the commutative diagram:

$$\begin{tikzcd} & \frac{GL(r,C)}{U^\pi(C^r)} \arrow[r, "{[i]}"] \arrow[d, "\Psi"] & \frac{GL(2r,\mathbb{R})}{O(r,r)} \arrow[d, "\tilde{\Psi}"] \\ M \arrow[ur, "\tilde{h}"] \arrow[dr, "h"'] & & \\ & \mathrm{Herm}(C^r) \arrow[r, "\mathcal{R}"] & \mathrm{Sym}_{r,r}(\mathbb{R}^{2r}), \end{tikzcd} \tag{3.6}$$

where $[i]$ is induced by the inclusion $i : GL(r,C) \hookrightarrow GL(2r,\mathbb{R})$. Since all other maps in the square of this diagram are totally geodesic, the map

$$\mathcal{R} : \mathrm{Herm}(C^r) \to \mathrm{Sym}_{r,r}(\mathbb{R}^{2r}), H \mapsto \mathbb{1}_{r,r}\iota(H)$$

is a totally geodesic map.
Notation: In the following work we use the notations $S(p,q) = GL(r,\mathbb{R})/O(p,q)$ and $H(r) = GL(r,C)/U^\pi(C^r)$.
Using the commutative diagram gives the proposition:

Proposition 4 *A map $h : M \to Herm(C^r)$ is para-pluriharmonic, if and only if $g = Re\, h : M \to Sym_{r,r}(\mathbb{R}^{2r})$ is para-pluriharmonic.*
A map $\tilde{h} : M \to H(r) = GL(r,C)/U^\pi(C^r)$ is para-pluriharmonic, if and only if $\tilde{g} = [i] \circ h : M \to S(r,r)$ is para-pluriharmonic.

Proof: As discussed in Lemma 1, 2 and 3 of this section the map $\mathcal{R} : \mathrm{Herm}(C^r) \to \mathrm{Sym}_{r,r}(\mathbb{R}^{2r})$ is totally geodesic and an immersion. This means that we are in the situation of proposition 1.
The second claim follows from the square of the commutative diagram (3.6) and from the information, that the composition of a map f from M to $\mathrm{Herm}(C^r)$ (respectively $\mathrm{Sym}_{r,r}(\mathbb{R}^{2r})$) with Ψ^{-1} (respectively $\tilde{\Psi}^{-1}$) is para-pluriharmonic, if and only if f is para-pluriharmonic. $\square$

4 Para-tt^*-bundles and associated para-pluriharmonic maps

In this section we recall the notion of (metric) para-tt^*-bundles and explain the correspondence between metric para-tt^*-bundles and para-pluriharmonic maps, which was given in [Sch2].

Definition 2 *A* para-tt^*-bundle *or* ptt^*-bundle (E, D, S) *over a para-complex manifold* (M, τ) *is a real vector bundle* $E \to M$ *endowed with a connection* D *and a section* $S \in \Gamma(T^*M \otimes \operatorname{End} E)$ *which satisfy the* ptt^*-equation

$$R^\theta = 0 \quad \textit{for all} \quad \theta \in \mathbb{R}\,, \tag{4.1}$$

where R^θ *is the curvature tensor of the connection* D^θ *defined by*

$$D^\theta_X := D_X + \cosh(\theta) S_X + \sinh(\theta) S_{\tau X} \quad \textit{for all} \quad X \in TM. \tag{4.2}$$

A metric ptt^*-bundle (E, D, S, g) *is a* ptt^**-bundle* (E, D, S) *endowed with a possibly indefinite* D*-parallel fiber metric* g *such that for all* $p \in M$

$$g(S_X Y, Z) = g(Y, S_X Z) \quad \textit{for all} \qquad X, Y, Z \in T_pM\,. \tag{4.3}$$

Remark 1 *1) If* (E, D, S) *is a* ptt^**-bundle then* (E, D, S^θ) *is a* ptt^**-bundle for all* $\theta \in \mathbb{R}$*, where* $S^\theta := D^\theta - D = \cosh(\theta) S + \sinh(\theta) S_\tau$*. The same remark applies to metric* ptt^**-bundles.*
2) The flatness of the connection D^θ *can be expressed in a set of equations on* D *and* S *which can be found in proposition 7 of [Sch2].*

Given a metric ptt^*-bundle (E, D, S, g), we consider the flat connection D^θ for a fixed $\theta \in \mathbb{R}$. Any D^θ-parallel frame $s = (s_1, \ldots, s_r)$ of E defines a map

$$G = G^{(s)} : M \to \operatorname{Sym}_{p,q}(\mathbb{R}^r); \quad x \mapsto G(x) := (g_x(s_i(x), s_j(x)))\,, \tag{4.4}$$

where (p, q) is the signature of the metric g.
Let G/K be a pseudo-Riemannian symmetric space with associated symmetric decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. We recall that a map $f : (M, \tau) \to G/K$ is said to be admissible, if the para-complex linear extension of its differential maps $T^{1,0}_x M$ (respectively $T^{0,1}_x M$) to an Abelian subspace of $\mathfrak{p}^C = \mathfrak{p} \otimes C$ for all $x \in M$.
If M is simply-connected then it was shown in [Sch2] Theorem 3, that $G : M \to Sym_{p,q}(\mathbb{R}^r)$ is para-pluriharmonic and that it induces an admissible para-pluriharmonic map $\tilde{G} : M \xrightarrow{G} Sym_{p,q}(\mathbb{R}^r) \xrightarrow{\tilde{\ }} S(p, q)$.
Conversely, we constructed in Theorem 4 of [Sch2] a metric tt^*-bundle ($E = M \times \mathbb{R}^{2r}, D = \partial - S, S = d\tilde{G}, g =< G\cdot, \cdot >_{\mathbb{R}^{2r}}$) over a simply-connected manifold from an admissible para-pluriharmonic map $\tilde{G} = \tilde{\Psi}^{-1} \circ G : M \to S(p, q)$. If M is not simply-connected, then we have to replace the maps G and $\tilde{G}$ by twisted para-pluriharmonic maps (cf. [Sch2] Theorems 5 and 6).

5 Para-harmonic bundles as solutions of *ptt**-geometries

In this section we introduce the notion of a para-harmonic bundle and show that every such bundle gives a solution of *ptt**-geometry.

Definition 3 *A* para-harmonic bundle $(E \to M, D, C, \bar{C}, h)$ *consists of the following data: A para-complex vector bundle E over a para-complex manifold (M, τ), a para-hermitian metric h, i.e. a smooth fiberwise para-hermitian sesquilinear scalar product, a metric connection D with respect to h and two C^∞-linear maps $C : \Gamma(E) \to \Gamma(\Lambda^{1,0}T^*M \otimes E)$ and $\bar{C} : \Gamma(E) \to \Gamma(\Lambda^{0,1}T^*M \otimes E)$, such that the connection*

$$D^{(\lambda)} = D + \lambda C + \bar{\lambda}\bar{C}$$

is flat for all $\lambda \in \tilde{\mathbb{S}}^1$ and $h(C_Z a, b) = h(a, \bar{C}_{\bar{Z}} b)$ with $a, b \in \Gamma(E)$ and $Z \in \Gamma(T^{1,0}M)$.

Theorem 1 *Let $(E \to M, D, C, \bar{C}, h)$ be a para-harmonic bundle over the para-complex manifold (M, τ), then $(E, D, S, g = Re\, h)$, with $S_X := C_Z + \bar{C}_{\bar{Z}}$ for $X = Z + \bar{Z} \in TM$ and $Z \in T^{1,0}M$, is a metric ptt*-bundle.*

Proof: For $\lambda = \cosh(\alpha) + e\sinh(\alpha) \in \tilde{\mathbb{S}}^1$ we compute $D^{(\lambda)}$:

$$\begin{aligned} D^{(\lambda)}_X &= D_X + \lambda C_Z + \bar{\lambda}\bar{C}_{\bar{Z}} = D_X + \cosh(\alpha)(C_Z + \bar{C}_{\bar{Z}}) + \sinh(\alpha)(eC_Z - e\bar{C}_{\bar{Z}}) \\ &= D_X + \cosh(\alpha)S_X + \sinh(\alpha)(C_{\tau Z} + \bar{C}_{\tau\bar{Z}}) \\ &= D_X + \cosh(\alpha)S_X + \sinh(\alpha)S_{\tau X} = D^\alpha_X. \end{aligned}$$

Hence we have

$$D^\alpha = D^{(\lambda)} \tag{5.1}$$

and D^α is flat if and only if $D^{(\lambda)}$ is flat.
Further we show, that S is g-symmetric. With $X = Z + \bar{Z}$ for $Z \in T^{1,0}M$ one finds

$$h(S_X\cdot,\cdot) = h(C_Z + \bar{C}_{\bar{Z}}\cdot,\cdot) = h(\cdot, C_Z + \bar{C}_{\bar{Z}}\cdot) = h(\cdot, S_X\cdot)$$

and consequently the symmetry of S with respect to $g = \mathrm{Re}h$. Finally a direct calculation shows $Dg = 0$. This proves, that $(E, D, S, g = \mathrm{Re}\, h)$ is a metric *ptt**-bundle. □

Remark: From the *ptt**-equations we know that $R^D + S \wedge S = 0$ and that $S \wedge S$ is of type $(1,1)$. Hence the $(0,2)$-part of R^D vanishes. By the para-complex generalization of a well-known theorem of complex geometry, the connection D induces a para-holomorphic structure such that D is adapted to this para-holomorphic structure (compare [LS] Proposition 2 and [E] for the case where M is a surface.). From $Dh = 0$ we obtain that D is the para-complex analogue of the Chern connection, which was introduced in [E].

6 The para-pluriharmonic maps associated to a para-harmonic bundle

In the last section we have shown, that every para-harmonic bundle induces a metric *ptt**-bundle and hence a para-pluriharmonic map to $S(r,r) = GL(2r,\mathbb{R})/O(r,r)$ where $r = \mathrm{rk}_C(E)$ is the para-complex rank of E. Afterwards, we use the additional information of the para-harmonic bundle structure to restrict the target of the para-pluriharmonic map to $H(r) = GL(r,C)/U^\pi(C^r)$. Finally we get a correspondence between para-harmonic bundles and admissible para-pluriharmonic maps into $H(r)$. Summarizing our current knowledge yields the corollary:

Corollary 1 *Let $(E \to M, D, C, \bar{C}, h)$ be a para-harmonic bundle over the simply connected para-complex manifold (M,τ), then the representation of $g = Re\, h$ in a $D^{(\lambda)}$-flat frame defines an admissible para-pluriharmonic map $\Phi_g : M \to S(r,r)$.*

Proof: This follows from the identity (5.1), i.e. $D^{(\lambda)}_X = D^\alpha_X$ for $\lambda = \cosh(\alpha) + e\sinh(\alpha) \in \tilde{\mathbb{S}}^1$ and from Theorem 3 of [Sch2]. $\square$

To restrict the image of Φ_g to $H(r)$ we analyze the real-part of h by using proposition 4.

Theorem 2 *Let $(E \to M, D, C, \bar{C}, h)$ be a para-harmonic bundle over the simply connected para-complex manifold (M,τ). Then the representation of h in a $D^{(\lambda)}$-flat frame defines a para-pluriharmonic map $\phi_h : M \to Herm(C^r)$ which itself induces an admissible para-pluriharmonic map $\tilde{\phi}_h = \Psi^{-1} \circ \phi_h : M \to H(r)$.*

Proof: The para-pluriharmonicity of ϕ_h and $\tilde{\phi}_h$ follows from corollary 1 and proposition 4. For the second part we observe, that the differential of $\mathcal{R} : \mathfrak{gl}_r(C) \to \mathfrak{gl}_{2r}(\mathbb{R})$ is a homomorphism of Lie-algebras and therefore preserves the vanishing of the Lie-bracket. $\square$

The following theorem gives the converse statement:

Theorem 3 *Let (M,τ) be a simply connected para-complex manifold and $E = M \times C^r$. An admissible para-pluriharmonic map $\tilde{\phi}_h : M \to H(r)$ induces an admissible para-pluriharmonic map $\tilde{\phi}_g = [i] \circ \tilde{\phi}_h : M \to S(r,r)$. Moreover, the admissible para-pluriharmonic map $\tilde{\phi}_h$ defines a para-harmonic bundle $(E, D = \partial - C - \bar{C}, C = (d\tilde{\phi}_h)^{1,0}, h = (\phi_h \cdot, \cdot)_{C^r})$, where ∂ is the para-complex linear extension on TM^C of the flat connection on $E = M \times C^r$.*

Proof: Thanks to proposition 4 the map $\tilde{\phi}_g$ is para-pluriharmonic. The same argument as in theorem 2 shows that $\tilde{\phi}_g$ is admissible. Therefore we obtain from Theorem 4 of [Sch2] a *ptt**-bundle $(E = M \times \mathbb{R}^{2r}, D = \partial - S, S = d\tilde{\phi}_g, g = < \phi_g \cdot, \cdot >_{\mathbb{R}^{2r}})$. Utilizing the additional information, which we have from the fact, that the map ϕ_g comes from ϕ_h, we show that $(E, D = \partial - C - \bar{C}, C = (d\tilde{\phi}_h)^{1,0}, h = (\phi_h \cdot, \cdot)_{C^r})$ is a para-harmonic bundle. The para-hermitian metric h is defined by

$$h = g + e\,\omega$$

with $\omega = g(E\cdot, \cdot)$. This is the standard relation between para-hermitian sesquilinear scalar products on para-complex vector spaces and the para-hermitian scalar products on the underlying real vector spaces.
It is $D_X E = [\partial_X - S_X, E] = [S_X, E] = 0$ with $X \in \Gamma(TM)$, since S is the differential of a map from M to $GL(r, C)$ and consequently commutes with E. Hence, $D\omega = 0$ follows from $Dg = 0$ and $Dh = 0$ from $D\omega = 0$ and $Dg = 0$.
The definition of S and S_τ in theorem 1, i.e. $S_X = C_Z + \bar{C}_{\bar{Z}}$ and $S_{\tau X} = C_{\tau Z} + \bar{C}_{\tau \bar{Z}}$ for $X = Z + \bar{Z}$ and $Z \in T^{1,0}M$, yields the definition of C by $2C_Z = S_X + ES_{\tau X}$ and $2C_{\bar{Z}} = S_X - ES_{\tau X}$. The identity $D^{(\lambda)}_X = D^\alpha_X$ for $\lambda = \cosh(\alpha) + e\sinh(\alpha) \in \tilde{\mathbb{S}}^1$ implies further the equivalence between the flatness of $D^{(\lambda)}$ and D^α.
We recall the relations $E^*g = -g$ and $(*)\, g(E\cdot, \cdot) = -g(\cdot, E\cdot)$, which implies the anti-symmetry of $\omega = g(E\cdot, \cdot)$ and $(*')\, \omega(E\cdot, \cdot) = -\omega(\cdot, E\cdot)$. Further we use the identities $(**)\, [S, E] = [S_\tau, E] = 0$ and that $(***)\, S, S_\tau$ are g-symmetric. Due to (**) and (***) we obtain: $(****)\, S, S_\tau$ ω-symmetric. These identities imply after a short computation $h(C_Z\cdot, \cdot) = h(\cdot, \bar{C}_{\bar{Z}}\cdot)$.
Using $S = -d\tilde{\phi}_g = -d([i] \circ \tilde{\phi}_h) = -d\tilde{\phi}_h$ we find extending S on TM^C to S^C for $Z \in T^{1,0}M$ the equations $C_Z = S^C_Z = -d\tilde{\phi}_h(Z)$ and $\bar{C}_{\bar{Z}} = -d\tilde{\phi}_h(\bar{Z})$. □

One can obtain similar results if the manifold (M, τ) is not simply connected. In fact, one has to replace the para-pluriharmonic maps by twisted para-pluriharmonic maps and to apply Theorems 5 and 6 of [Sch2] instead of Theorems 3 and 4 of the same reference.

References

[B] W. Bertram, *The Geometry of Jordan and Lie-Structures,* Springer Lecture Notes in Mathematics vol. **1754.**

[CV] S. Cecotti and C. Vafa, *Topological-antitopological fusion,* Nuclear Physics **B 367** (1991), 351-461.

[CMMS] V. Cortés, C. Mayer, T. Mohaupt and F. Saueressig, *Special Geometry of Euclidean Supersymmetry I: Vectormultiplets*, J. High Energy Phys. JHEP03 (2004) 028, hep-th/0312001 2004.

[CS] V. Cortés and L. Schäfer, *Topological-antitopological fusion equations, pluriharmonic maps and special Kähler manifolds*, Progress in Mathematics 234, Birkhäuser 2005.

[CFG] V. Cruceanu and P. Fortuny, P. M. Gadea, *A survey on paracomplex geometry,* Rocky Mt. J. Math. **26** (1996), No.1, 83-115.

[ES] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds,* Am. J. Math. **86** (1964), 109-160.

[E] S. Erdem, *Paraholomorphic structures and the connections of vector bundles over paracomplex manifolds*, New Zealand J. of Math., Vol. **30** (2001), no.1, 41-50.

[EST] F. Etayo, R. Santamaria and U. R. Trias, *The geometry of a Bi-Lagrangian manifold*, Differ. Geom. Appl. **24** (2006), No. 1, 33-59.

[LS] M.-A. Lawn and L. Schäfer, *Decompositions of para-complex vector-bundles and affine para-complex immersions,* Res. in Math. vol. **48**, 3/4.

[L] P. Libermann, *Sur les structures presque paracomplexes,* C. R. Acad. Sci., Paris **234** (1952), 2517-2519.

[Sch1] L. Schäfer, *tt*-bundles and pluriharmonic maps*, Ann. Global Anal. Geom. 28 (2005) no. 3, 285-300.

[Sch2] L. Schäfer, *tt^*-bundles in para-complex geometry, special para-Kähler manifolds and para-pluriharmonic maps*, Diff. Geom. and Appl. **24** no.1 (2006), 60-89.

[Sch3] L. Schäfer, *Harmonic bundles, topological-antitoplological fusion and the related puriharmonic maps*, J. Geom. Phys. **56** (2006), no. 5, 830–842.

[Si] C.T. Simpson: Higgs-bundles and local systems, *Institut des hautes Etudes scientifiques - Publication mathematiques,* **75** *(1992)*, 5-95.